

Entropic bounds and continual measurements

Alberto Barchielli

Politecnico di Milano, Dipartimento di Matematica,
Piazza Leonardo da Vinci 32, I-20133 Milano, Italy.
E-mail: Alberto.Barchielli@polimi.it

Giancarlo Lupieri

Università degli Studi di Milano, Dipartimento di Fisica,
Via Celoria 16, I-20133 Milano, Italy.
E-mail: Giancarlo.Lupieri@mi.infn.it

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Abstract

Some bounds on the entropic informational quantities related to a quantum continual measurement are obtained and the time dependencies of these quantities are studied.

1 Introduction

In the problem of information transmission through quantum systems, various entropic quantities appear which characterize the performances of the encoding and decoding apparatuses. Due to the peculiar character of a quantum measurement, many bounds on the informational quantities involved have been proved to hold [1–8]. In the case of measurements continual in time, these bounds acquire new aspects (family of measurements are now involved) and new problems arise. A typical question is about which of the various entropic measures of information is monotonically increasing or decreasing in time. We already started the study of this subject in Refs. [9, 10]; here we apply to the case of continual measurements the new techniques developed [6–8] for the time independent case.

1.1 Notations and preliminaries

We denote by $\mathcal{L}(\mathcal{A}; \mathcal{B})$ the space of bounded linear operators from \mathcal{A} to \mathcal{B} , where \mathcal{A}, \mathcal{B} are Banach spaces; moreover we set $\mathcal{L}(\mathcal{A}) := \mathcal{L}(\mathcal{A}; \mathcal{A})$.

Let \mathcal{H} be a separable complex Hilbert space; a normal state on $\mathcal{L}(\mathcal{H})$ is identified with a statistical operator, $\mathcal{T}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ are the trace-class and the space of the statistical operators on \mathcal{H} , respectively, and $\|\rho\|_1 := \text{Tr} \sqrt{\rho^* \rho}$, $\langle \rho, a \rangle := \text{Tr}_{\mathcal{H}} \{\rho a\}$, $\rho \in \mathcal{T}(\mathcal{H})$, $a \in \mathcal{L}(\mathcal{H})$.

More generally, if a belongs to a W^* -algebra and ρ to its dual \mathcal{M}^* or predual \mathcal{M}_* , the functional ρ applied to a is denoted by $\langle \rho, a \rangle$.

1.1.1 A quantum/classical algebra

Let (Ω, \mathcal{F}, Q) be a measure space, where Q is a σ -finite measure. By Theorem 1.22.13 of [11], the W^* -algebra $L^\infty(\Omega, \mathcal{F}, Q) \otimes \mathcal{L}(\mathcal{H})$ (W^* -tensor product) is naturally isomorphic to the W^* -algebra $L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H}))$ of all the $\mathcal{L}(\mathcal{H})$ -valued Q -essentially bounded weakly* measurable functions on Ω . Moreover ([11], Proposition 1.22.12), the predual of this W^* -algebra is $L^1(\Omega, \mathcal{F}, Q; \mathcal{T}(\mathcal{H}))$, the Banach space of all the $\mathcal{T}(\mathcal{H})$ -valued Bochner Q -integrable functions on Ω , and this predual is naturally isomorphic to $L^1(\Omega, \mathcal{F}, Q) \otimes \mathcal{T}(\mathcal{H})$ (tensor product with respect to the greatest cross norm — [11], pp. 45, 58, 59, 67, 68).

Let us note that a normal state σ on $L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H}))$ is a measurable function $\omega \mapsto \sigma(\omega) \in \mathcal{T}(\mathcal{H})$, $\sigma(\omega) \geq 0$, such that $\text{Tr}_{\mathcal{H}}\{\sigma(\omega)\}$ is a probability density with respect to Q .

1.2 Quantum channels and entropies

1.2.1 Relative and mutual entropies

The general definition of the relative entropy $S(\Sigma|\Pi)$ for two states Σ and Π is given in [12]; here we give only some particular cases of the general definition.

Let us consider two quantum states $\sigma, \tau \in \mathcal{S}(\mathcal{H})$ and two classical states q_k on $L^\infty(\Omega, \mathcal{F}, Q)$ (two probability densities with respect to Q). The quantum relative entropy and the classical one are

$$S_q(\sigma|\tau) = \text{Tr}_{\mathcal{H}}\{\sigma(\log \sigma - \log \tau)\}, \quad (1a)$$

$$S_c(q_1|q_2) = \int_{\Omega} Q(d\omega) q_1(\omega) \log \frac{q_1(\omega)}{q_2(\omega)}. \quad (1b)$$

We shall need also the von Neumann entropy of a state $\tau \in \mathcal{S}(\mathcal{H})$: $S_q(\tau) := -\text{Tr}\{\tau \log \tau\}$.

Let us consider now two normal states σ_k on $L^\infty(\Omega, \mathcal{F}, Q; \mathcal{L}(\mathcal{H}))$ and set $q_k(\omega) := \text{Tr}\{\sigma_k(\omega)\}$, $\varrho_k(\omega) := \sigma_k(\omega)/q_k(\omega)$ (these definitions hold where the denominators do not vanish and are completed arbitrarily where the denominators vanish). Then, the relative entropy is

$$S(\sigma_1|\sigma_2) = \int_{\Omega} Q(d\omega) \text{Tr}_{\mathcal{H}}\{\sigma_1(\omega)(\log \sigma_1(\omega) - \log \sigma_2(\omega))\} \quad (2a)$$

$$= S_c(q_1|q_2) + \int_{\Omega} Q(d\omega) q_1(\omega) S_q(\varrho_1(\omega)|\varrho_2(\omega)). \quad (2b)$$

We are using a subscript “c” for classical entropies, a subscript “q” for purely quantum ones and no subscript for general entropies, eventually of a mixed character.

Classically a mutual entropy is the relative entropy of a joint probability with respect to the product of its marginals and this key notion can be generalized immediately to states on von Neumann algebras, every times we have a state on a tensor product of algebras [6–8].

1.2.2 Channels

Definition 1. ([12] p. 137) Let \mathcal{M}_1 and \mathcal{M}_2 be two W^* -algebras. A linear map Λ^* from \mathcal{M}_2 to \mathcal{M}_1 is said to be a *channel* if it is completely positive, unital (i.e. identity preserving) and normal (or, equivalently, weakly* continuous).

Due to the equivalence [13] of w^* -continuity and existence of a preadjoint Λ , a *channel* is equivalently defined by: Λ is a completely positive linear map from the predual \mathcal{M}_{1*} to the predual \mathcal{M}_{2*} , normalized in the sense that $\langle \Lambda[\rho], \mathbb{1}_2 \rangle_2 = \langle \rho, \mathbb{1}_1 \rangle_1$, $\forall \rho \in \mathcal{M}_{1*}$. Let us note also that Λ maps normal states on \mathcal{M}_1 into normal states on \mathcal{M}_2 .

A key result which follows from the convexity properties of the relative entropy is *Uhlmann monotonicity theorem* ([12], Theor. 1.5 p. 21), which implies that channels decrease the relative entropy.

Theorem 1. If Σ and Π are two normal states on \mathcal{M}_1 and Λ^* is a channel from $\mathcal{M}_2 \rightarrow \mathcal{M}_1$, then $S(\Sigma|\Pi) \geq S(\Lambda[\Sigma]|\Lambda[\Pi])$.

1.3 Continual measurements

Let us axiomatize the properties of a probability space where an independent-increment process lives and that ones of the σ -algebras generated by its increments. The probability measure Q_1 we are introducing will play the role of a reference measure.

Assumption 1. Let (X, \mathcal{X}, Q_1) be a probability space with (X, \mathcal{X}) standard Borel. Moreover:

1. $\{\mathcal{X}_t^s, 0 \leq s \leq t\}$ is a two-times filtration of sub- σ -algebras: $\mathcal{X}_t^s \subset \mathcal{X}_T^r \subset \mathcal{X}$ for $0 \leq r \leq s \leq t \leq T$;
2. $\forall t \geq 0$, \mathcal{X}_t^t is trivial;
3. $\mathcal{X}_t^s = \bigwedge_{T>t} \mathcal{X}_T^s$ for $0 \leq s \leq t$;
4. $\mathcal{X}_t^s = \bigvee_{r:s<r<t} \mathcal{X}_t^r$ for $0 \leq s < t$;
5. $\mathcal{X} = \bigvee_{t>0} \mathcal{X}_t^0$;
6. for $0 \leq r \leq s \leq t \leq T$, \mathcal{X}_s^r and \mathcal{X}_T^t are Q_1 -independent.

Continual measurements are a quantum analog of classical processes with independent increments [10, 14]. As any kind of quantum measurement, a continual measurement is represented by *instruments* [15–17], but, as shown in [7], instruments are equivalent to particular types of channels. Here we introduce continual measurements directly as a family of channels satisfying a set of axioms (cf. also [10, 18]).

Assumption 2. Let \mathcal{H} be a separable complex Hilbert space. For all s, t , $0 \leq s \leq t$, we have a channel

$$\tilde{\Lambda}_t^s : L^1(X, \mathcal{X}_s^0, Q_1; \mathcal{T}(\mathcal{H})) \rightarrow L^1(X, \mathcal{X}_t^0, Q_1; \mathcal{T}(\mathcal{H}))$$

such that

1. $\tilde{\Lambda}_t^t = \mathbb{1}$, $t \geq 0$;
2. $\tilde{\Lambda}_t^s \circ \tilde{\Lambda}_s^r = \tilde{\Lambda}_t^r$, $0 \leq r \leq s \leq t$;
3. $\forall \eta \in \mathcal{T}(\mathcal{H})$, $\tilde{\Lambda}_t^s[\eta]$ is \mathcal{X}_t^s -measurable, $0 \leq s \leq t$;
4. $\forall \eta \in \mathcal{T}(\mathcal{H})$, $\forall q \in L^1(X, \mathcal{X}_s^0, Q_1)$, $\tilde{\Lambda}_t^s[q\eta] = q\tilde{\Lambda}_t^s[\eta]$, $0 \leq s \leq t$, (i.e. $\tilde{\Lambda}_t^s[q\eta](x) = q(x)\tilde{\Lambda}_t^s[\eta](x)$ a.s.).

By points (3), (4) of Assumption 2 and (6) of Assumption 1, one gets:
 $\forall \sigma_s \in L^1(X, \mathcal{X}_s^0, Q_1; \mathcal{T}(\mathcal{H}))$, $0 \leq s \leq t$,

$$\mathbb{E}_{Q_1} [\tilde{\Lambda}_t^s[\sigma_s] | \mathcal{X}_t^s] = \tilde{\Lambda}_t^s [\mathbb{E}_{Q_1} [\sigma_s]]. \quad (3)$$

Here \mathbb{E}_{Q_1} and $\mathbb{E}_{Q_1}[\bullet | \mathcal{X}_t^s]$ are the classical expectation and conditional expectation extended to operator-valued random variable.

Let us also define the *evolution*

$$\mathcal{U}(t, s)[\tau] := \mathbb{E}_{Q_1} [\tilde{\Lambda}_t^s[\tau]], \quad \tau \in \mathcal{T}(\mathcal{H}), \quad 0 \leq s \leq t; \quad (4)$$

$\mathcal{U}(t, s)$ is a channel from $\mathcal{T}(\mathcal{H})$ into $\mathcal{T}(\mathcal{H})$. By points (2), (3), (4) of Assumption 2, for $0 \leq r \leq s \leq t$, $\sigma_s \in L^1(X, \mathcal{X}_s^0, Q_1; \mathcal{T}(\mathcal{H}))$, we get

$$\mathcal{U}(t, s) \circ \mathcal{U}(s, r) = \mathcal{U}(t, r), \quad \mathbb{E}_{Q_1} [\tilde{\Lambda}_t^s[\sigma_s] | \mathcal{X}_s^0] = \mathcal{U}(t, s)[\sigma_s]. \quad (5)$$

The quantum continual measurements is represented by the operators $\tilde{\Lambda}_t^s$, in the sense that they give probabilities and state changes. If $\eta_0 \in \mathcal{S}(\mathcal{H})$ is the initial state at time 0 and $B \in \mathcal{X}_t^0$ is any event involving the output in the interval $(0, t)$, then $\int_B \text{Tr}\{\tilde{\Lambda}_t^0[\eta_0](x)\} Q_1(dx)$ is the probability of the event B and $\frac{\tilde{\Lambda}_t^0[\eta_0](x)}{\text{Tr}\{\tilde{\Lambda}_t^0[\eta_0](x)\}}$ is the state at time t , conditional on the result x (the *a posteriori* state). Instead, $\mathcal{U}(t, 0)[\eta_0]$ represents the state of the system at time t , when the results of the measurement are not taken into account (the *a priori* state).

2 The initial state and the measurement

2.1 Ensembles

In quantum information theory, not only single states are used, but also families of quantum states with a probability law on them, called ensembles. An *ensemble* $\{\mu, \rho\}$ is a probability measure $\mu(dy)$ on some measurable space (Y, \mathcal{Y}) together with a random variable $\rho : Y \rightarrow \mathcal{S}(\mathcal{H})$. Alternatively, an ensemble can be seen as a quantum/classical state of the type described in Section 1.1.1. Given an ensemble, one can introduce an *average state* $\bar{\rho} \in \mathcal{S}(\mathcal{H})$

$$\bar{\rho} := \mathbb{E}_\mu[\rho] = \int_Y \mu(dy) \rho(y); \quad (6)$$

the integrals involving trace class operators are always understood as Bochner integrals. Finally, the average relative entropy of the states $\rho(y)$ with respect to $\bar{\rho}$ is called the “ χ -quantity” of the ensemble:

$$\chi\{\mu, \rho\} := \int_Y \mu(dy) S_q(\rho(y) | \bar{\rho}) = \mathbb{E}_\mu [S_q(\rho | \bar{\rho})]. \quad (7)$$

This new quantity plays an important role in the whole quantum information theory [3, 20] and can be thought as a measure of some kind of quantum information stored in the ensemble.

2.2 The letter states

Let us consider the typical setup of quantum communication theory. A message is transmitted by encoding the letters in some quantum states, which are possibly corrupted by a quantum noisy channel; at the end of the channel the receiver attempts to decode the message by performing measurements on the quantum system. So, one has an alphabet A and the letters $\alpha \in A$ are transmitted with some a priori probabilities P_i . Each letter α is encoded in a quantum state and we denote by $\rho_i(\alpha)$ the state associated to the letter α as it arrives to the receiver, after the passage through the transmission channel. While it is usual to consider a finite alphabet, also general continuous parameter spaces are acquiring importance [19, 20].

Assumption 3. Let (A, \mathcal{A}, Q_0) be a probability space with (A, \mathcal{A}) standard Borel and let σ_i be a normal state on $L^\infty(A, \mathcal{A}, Q_0; \mathcal{L}(\mathcal{H}))$.

Let us set

$$q_i(\alpha) := \text{Tr}\{\sigma_i(\alpha)\}, \quad \rho_i(\alpha) := \frac{\sigma_i(\alpha)}{q_i(\alpha)}, \quad P_i(d\alpha) := q_i(\alpha)Q_0(d\alpha); \quad (8)$$

q_i is a probability density and $\{P_i, \rho_i\}$ is the initial *ensemble*. The average state and the χ -quantity of the initial ensemble are

$$\eta_0 := \mathbb{E}_{Q_0}[\sigma_i] = \int_A P_i(d\alpha) \rho_i(\alpha), \quad (9)$$

$$\chi\{P_i, \rho_i\} := \int_A P_i(d\alpha) S_q(\rho_i(\alpha) | \eta_0). \quad (10)$$

The quantity $\chi\{P_i, \rho_i\}$ is known also as Holevo capacity [3, 20].

2.3 Probabilities and states derived from η_0

For $0 \leq r \leq s \leq t$ we define:

$$\eta_t := \mathcal{U}(t, 0)[\eta_0], \quad \tilde{\sigma}_t^r := \tilde{\Lambda}_t^r[\eta_r], \quad \tilde{q}_t^s := \|\tilde{\sigma}_t^s\|_1, \quad \tilde{\varrho}_t^r := \frac{\tilde{\sigma}_t^r}{\tilde{q}_t^r}. \quad (11)$$

Then, η_t and $\varrho_t^r(x)$ are states on $\mathcal{L}(\mathcal{H})$, \tilde{q}_t^s is a state on $L^\infty(X, \mathcal{X}_t^0, Q_1)$ and $\tilde{\sigma}_t^r$ a state on $L^\infty(X, \mathcal{X}_t^0, Q_1; \mathcal{L}(\mathcal{H}))$. We have also

$$\mathbb{E}_{Q_1}[\tilde{q}_t^r | \mathcal{X}_s^0] = \tilde{q}_s^r, \quad \mathbb{E}_{Q_1}[\tilde{q}_t^r | \mathcal{X}_t^s] = \tilde{q}_t^s. \quad (12)$$

Moreover, there exists a unique probability P_1 on (X, \mathcal{X}) such that $P_1(dx)|_{\mathcal{X}_t^0} = \tilde{q}_t^0(x)Q_1(dx)$ for all $t \geq 0$. Also $P_1(dx)|_{\mathcal{X}_t^s} = \tilde{q}_t^s(x)Q_1(dx)$ holds.

2.4 The general setup

It is useful to unify the initial distribution and the distribution of the measurement results in a unique filtered probability space. Let us set:

$$\Omega := A \times X, \quad \omega := (\alpha, x), \quad \pi_0(\omega) := \alpha, \quad \pi_1(\omega) := x, \quad (13a)$$

$$\sigma_0 := \sigma_i \circ \pi_0, \quad q_0 := q_i \circ \pi_0 = \|\sigma_0\|_1, \quad \rho_0 := \rho_i \circ \pi_0 = \frac{\sigma_0}{\|\sigma_0\|_1}, \quad (13b)$$

$$\mathcal{F} := \mathcal{A} \otimes \mathcal{X}, \quad Q := Q_0 \otimes Q_1, \quad (13c)$$

$$\mathcal{F}_0 := \{B \times X : B \in \mathcal{A}\}, \quad \mathcal{F}_t^s := \{A \times Y : Y \in \mathcal{X}_t^s\}, \quad (13d)$$

$$\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{F}_t^0 = \sigma\{B \times Y : B \in \mathcal{A}, Y \in \mathcal{X}_t^0\}, \quad (13e)$$

By defining $\Lambda_t^s := \mathbb{1} \otimes \tilde{\Lambda}_t^s$, we extend $\tilde{\Lambda}_t^s$ to $L^1(\Omega, \mathcal{F}_s, Q; \mathcal{T}(\mathcal{H})) \simeq L^1(A, \mathcal{A}, Q_0) \otimes L^1(X, \mathcal{X}_s^0, Q_1; \mathcal{T}(\mathcal{H}))$. Similarly, we extend $\mathcal{U}(t, s)$ to $L^1(\Omega, \mathcal{F}_s, Q; \mathcal{T}(\mathcal{H})) \simeq L^1(\Omega, \mathcal{F}_s, Q) \otimes \mathcal{T}(\mathcal{H})$. Let us also set:

$$\sigma_t := \Lambda_t^0[\sigma_0], \quad \sigma_t^s := \tilde{\sigma}_t^s \circ \pi_1 = \Lambda_t^s[\eta_s], \quad q_t := \|\sigma_t\|_1, \quad (14a)$$

$$q_t^s := \tilde{q}_t^s \circ \pi_1 = \|\sigma_t^s\|_1, \quad \rho_t := \frac{\sigma_t}{\|\sigma_t\|_1}, \quad \varrho_t^s := \tilde{\varrho}_t^s \circ \pi_1 = \frac{\sigma_t^s}{\|\sigma_t^s\|_1}. \quad (14b)$$

In the computations of the following sections we shall need various properties of the quantities we have just introduced; here we summarize such properties. Let r, s, t be three ordered times: $0 \leq r \leq s \leq t$. Then, σ_t and σ_t^s are states on $L^\infty(\Omega, \mathcal{F}_t, Q; \mathcal{L}(\mathcal{H}))$ and

$$\mathbb{E}_Q[q_t | \mathcal{F}_s] = q_s, \quad \mathbb{E}_Q[q_t | \mathcal{F}_t^s] = \mathbb{E}_Q[q_t^r | \mathcal{F}_t^s] = q_t^s, \quad (15a)$$

$$\mathbb{E}_Q[q_t^r | \mathcal{F}_s] = q_s^r, \quad \mathbb{E}_Q[\sigma_t | \mathcal{F}_t^s] = \mathbb{E}_Q[\sigma_t^r | \mathcal{F}_t^s] = \sigma_t^s, \quad (15b)$$

$$\mathbb{E}_Q[\sigma_t | \mathcal{F}_s] = \mathcal{U}(t, s)[\sigma_s], \quad \mathbb{E}_Q[\sigma_t^r | \mathcal{F}_s] = \mathcal{U}(t, s)[\sigma_s^r], \quad (15c)$$

$$\mathbb{E}_Q[\sigma_t^s | \mathcal{F}_s] = \eta_t, \quad \eta_t = \mathbb{E}_Q[\sigma_t], \quad \sigma_t = \Lambda_t^s[\sigma_s], \quad (15d)$$

$$\sigma_t^r = \Lambda_t^s[\sigma_s^r], \quad \frac{\Lambda_t^s[\rho_s]}{\|\Lambda_t^s[\rho_s]\|_1} = \rho_t, \quad \frac{\Lambda_t^s[\varrho_s^r]}{\|\Lambda_t^s[\varrho_s^r]\|_1} = \varrho_t^r. \quad (15e)$$

We have that $\{q_t, t \geq 0\}$ is a non-negative, mean one, Q -martingale. Then, there exists a unique probability P on (Ω, \mathcal{F}) such that $\forall t \geq 0$

$$P(d\omega)|_{\mathcal{F}_t} = q_t(\omega)Q(d\omega). \quad (16)$$

Moreover,

$$P(d\alpha \times X) = P_1(d\alpha), \quad P(A \times dx) = P_1(dx), \quad (17)$$

$$P(d\omega)|_{\mathcal{F}_t^s} = q_t^s(\omega)Q(d\omega), \quad \eta_t = \mathbb{E}_P[\rho_t] = \mathcal{U}(t, s)[\eta_s]. \quad (18)$$

3 Mutual entropies and informational bounds

Here and in the following we shall have always $0 \leq u \leq r \leq s \leq t$.

3.1 The state q_t and the classical information

Let us consider the state q_t and its marginals $\mathbb{E}_Q[q_t|\mathcal{F}_r] = q_r$, $\mathbb{E}_Q[q_t|\mathcal{F}_t^r] = q_t^r$. Then, we can introduce the classical mutual entropy:

$$S_c(q_t|q_r q_t^r) = \int_{\Omega} P(d\omega) \log \frac{q_t(\omega)}{q_r(\omega)q_t^r(\omega)} =: I_c(r, t). \quad (19a)$$

Note that $I_c(t, t) = 0$. For $r = 0$ we have the input/output classical information gain:

$$I_c(0, t) = S_c(q_t|q_i \otimes \tilde{q}_t^0) \equiv \int_{A \times X} P(d\alpha \times dx) \log \frac{q_t(\alpha, x)}{q_i(\alpha)\tilde{q}_t^0(x)}. \quad (19b)$$

By applying the monotonicity theorem and the channel $\mathbb{E}_Q[\bullet|\mathcal{F}_s]$ to the couple of states q_t and $q_r q_t^r$, we get

$$S_c(q_t|q_r q_t^r) \geq S_c(\mathbb{E}_Q[q_t|\mathcal{F}_s]|\mathbb{E}_Q[q_r q_t^r|\mathcal{F}_s]) = S_c(q_s|q_r q_s^r), \quad (20)$$

which becomes

$$I_c(r, t) \geq I_c(r, s). \quad (21)$$

The function $t \mapsto I_c(s, t)$ is non decreasing.

3.2 The state σ_s and the main bound

A useful quantity, with the meaning of a measure of the “quantum information” left in the a posteriori states, is the mean χ -quantity

$$\overline{\chi}(s, t) := \int_{\Omega} P(d\omega) S_q(\rho_t(\omega)|\varrho_t^s(\omega)) = \mathbb{E}_P [S_q(\rho_t|\varrho_t^s)]. \quad (22)$$

The interpretation as a mean χ -quantity is due to the fact that $\overline{\chi}(s, t) = \mathbb{E}_P [\mathbb{E}_P [S_q(\rho_t|\varrho_t^s)|\mathcal{F}_t^s]]$. But by Eq. (7) and $\mathbb{E}_P[\rho_t|\mathcal{F}_t^s] = \varrho_t^s$, $\mathbb{E}_P[\varrho_t^s|\mathcal{F}_t^s] = \varrho_t^s$, we have that $\mathbb{E}_P [S_q(\rho_t|\varrho_t^s)|\mathcal{F}_t^s]$ is a random χ -quantity. Note that

$$\overline{\chi}(t, t) = \int_{\Omega} P(d\omega) S_q(\rho_t(\omega)|\eta_t) =: \chi\{P, \rho_t\}. \quad (23)$$

Let us consider the state σ_s and its marginals $\mathbb{E}_Q[\text{Tr}\{\sigma_s\}|\mathcal{F}_r] = q_r$, $\mathbb{E}_Q[\sigma_s|\mathcal{F}_s^r] = \sigma_s^r$. Then, we have the mutual entropy

$$S(\sigma_s|q_r \sigma_s^r) = I_c(r, s) + \overline{\chi}(r, s). \quad (24)$$

For $r = s$ and for $r = s = 0$ this equation reduces to

$$S(\sigma_s|q_s \eta_s) = \chi\{P, \rho_s\}, \quad S(\sigma_0|q_0 \eta_0) = \chi\{P, \rho_0\} = \chi\{P, \rho_i\}. \quad (25)$$

By applying the monotonicity theorem and the channel Λ_t^s to the couple of states σ_s and $q_r \sigma_s^r$, we get

$$S(\sigma_s|q_r \sigma_s^r) \geq S(\Lambda_t^s[\sigma_s]|\Lambda_t^s[q_r \sigma_s^r]) = S(\sigma_t|q_r \sigma_t^r), \quad (26)$$

which becomes

$$\overline{\chi}(r, s) - \overline{\chi}(r, t) \geq I_c(r, t) - I_c(r, s) \geq 0. \quad (27)$$

Therefore, the function $t \mapsto \bar{\chi}(s, t)$ is non increasing.

For $r = s$ we get

$$S(\sigma_s | q_s \eta_s) \geq S(\sigma_t | q_s \sigma_t^s), \quad (28)$$

which gives the upper bound for I_c :

$$0 \leq I_c(s, t) \leq \chi\{P, \rho_s\} - \bar{\chi}(s, t). \quad (29)$$

For $s = r = 0$, it reduces to

$$0 \leq I_c(0, t) \leq \chi\{P_i, \rho_i\} - \int_{A \times X} P(d\alpha \times dx) S_q(\rho_t(\alpha, x) | \tilde{\varrho}_t^0(x)). \quad (30)$$

The bound (30) is the translation in terms of continual measurements of the bound of Section 3.3.4 of [7], which in turn is a generalization of a bound by Schumacher, Westmoreland and Wootters [5]. Equation (30) is a strengthening of the Holevo bound [3] $I_c(0, t) \leq \chi\{P_i, \rho_i\}$.

3.3 Quantum information gain

Let us consider now the *quantum information gain* defined by the quantum entropy of the pre-measurement state minus the mean entropy of the a posteriori states [1, 2, 4]. It is a measure of the gain in purity (or loss, if negative) in passing from the pre-measurement state to the post-measurement a posteriori states. In the continual case, we can consider the quantum information gain in the time interval (s, t) when the system is prepared in the ensemble $\{P_i, \rho_i\}$ at time 0 or when it is prepared in the state η_r at time r :

$$I_q(s, t) := \int_{\Omega} P(d\omega) [S_q(\rho_s(\omega)) - S_q(\rho_t(\omega))], \quad (31a)$$

$$I_q(r; s, t) := \int_{\Omega} P(d\omega) [S_q(\varrho_s^r(\omega)) - S_q(\varrho_t^r(\omega))]. \quad (31b)$$

By this definition we have immediately

$$I_q(r, t) = I_q(r, s) + I_q(s, t), \quad I_q(u; r, t) = I_q(u; r, s) + I_q(u; s, t). \quad (32)$$

It has been proved [4] that the quantum information gain is positive for all initial states if and only if the measurement sends pure initial states into pure a posteriori states.

As in the single time case [6–8], inequality (27) can be easily transformed into an inequality involving I_q :

$$I_q(r; s, t) - I_q(s, t) \geq I_c(r, t) - I_c(r, s) \geq 0. \quad (33)$$

Let us take an initial ensemble made up of pure states: $\rho_i(\alpha)^2 = \rho_i(\alpha)$, $\forall \alpha \in A$. Let us assume that the continual measurement preserve pure states: the states $\rho_t(\alpha, x)$ are pure for all choices of t, α, x . Then, the von Neumann entropy of $\rho_t(\omega)$ vanishes and we have $I_q(s, t) = 0$ for all choices of s and t . From the second of Eqs. (32) and Eq. (33) we get

$$I_q(u; r, t) - I_q(u; r, s) = I_q(u; s, t) \geq I_c(u, t) - I_c(u, s) \geq 0, \quad (34)$$

i.e. the function $t \mapsto I_q(u; r, t)$ is non decreasing for “pure” continual measurements.

In particular, by taking $u = r = 0$ we have

$$I_q(0; 0, t) = S_q(\eta_0) - \int_X P_1(dx) S_q(\tilde{\rho}_t^0(x)). \quad (35)$$

For a continual measurement sending every pure initial state into pure a posteriori states, $\forall \eta_0 \in \mathcal{S}(\mathcal{H})$ the quantum information gain $I_q(0; 0, t)$ is non negative, non decreasing in time and with $I_q(0; 0, 0) = 0$.

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